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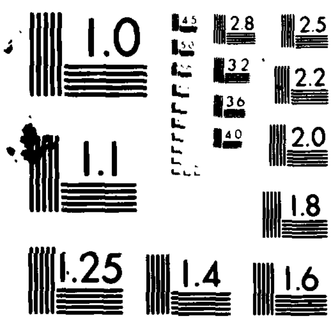

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NOMOGRAPHIC FUNCTIONS ARE NOWHERE DENSE

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ABSTRACT

A nomographic function of k variables is one that can be represented by the format

$$f(x_1, x_2, \ldots, x_k) = h(\phi_1(x_1) + \phi_2(x_2) + \ldots + \phi_k(x_k))$$

where the ϕ_i and h are continuous. Any individual nomographic function is very special in nature, since it is constructed from functions of one variable and addition alone. However, Kolmogorov showed in 1957 that every continuous function of k variables has a representation as a sum of not more than $2k+1$ nomographic functions. The present paper throws additional light on this, and settles a conjecture, by giving a constructive proof that the nomographic functions form a nowhere dense subset of the space $C[I^k]$ of continuous real valued functions on the k-cell.

AMS (MOS) Subject Classifications: 41A30, 26A72.

Key Words: approximations of functions of several variables; superpositions; nowhere dense sets

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SIGNIFICANCE AND EXPLANATION

An important topic in approximation theory is the study of ways to approximate complicated functions of many variables by combinations of simpler functions. One important type of the latter are the nomographic functions, which can be written entirely in terms of addition and functions of one variable. The present paper shows that these are inherently a very sparse subset of the class of all continuous functions; this places a severe limitation upon their use as single functions, but not if they are added together.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NOMOGRAPHIC FUNCTIONS ARE NOWHERE DENSE

R. Creighton Buck

Let $I = [-1, 1]$. The class $\mathcal{N}^k$ of nomographic functions on the k-cube I^k are those that can be represented there in the special format

(1) $$f(x) = f(x_1, x_2, \dots, x_k)$$
$$= h(\phi_1(x_1) + \phi(x_2) + \dots + \phi_k(x_k))$$

where the ϕ_i are real valued continuous functions on I and h is continuous on $-\infty < t < \infty$. Interest in $\mathcal{N}^k$ revived when Kolmogorov used them in 1957 to settle Hilbert's 13th problem by showing that every continuous real function on I^k could be written as a sum of $2k+1$ nomographic functions. [5], [6]

An individual nomographic function is quite special. For example, if $f \in \mathcal{N}^2$ and the component functions h, ϕ_1 and ϕ_2 are at least of class C^3, then f must be a solution of a specific third order nonlinear PDE, characteristic for the class $\mathcal{N}^2$. (See [3].) The conjecture is that in general $\mathcal{N}^k$ is a rather sparse subset of the banach space $C[I^k]$ of all real valued continuous functions on I^k; in the present paper, we present a constructive proof that $\mathcal{N}^k$ is nowhere dense.

This fact does not conflict with the Kolmogorov property of the set $\mathcal{N}^k$; for example, the interval $[0,1]$ can be written easily as the algebraic sum of two (or m for any $m \geq 2$) copies of a nowhere dense subset E.

Our proof follows a familiar pattern. To make this explicit we give a general approach to such proofs, and then verify later that the special property used in the proof holds for $\mathcal{N}^k$.

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Let D be a compact set in R^n with non void interior and $C[D]$ the space of real valued continuous functions F on D, with the usual uniform norm

$$\|F\|_D = \max_{p \in D} |F(p)| \quad .$$

Let $\mathcal{F}$ be a subset of $C[D]$ which we wish to prove nowhere dense. The key property used is the existence of special functions in $C[D]$ that fail to belong, locally, to the closure of $\mathcal{F}$.

Theorem 1. Suppose that it is true that there is a point p_o interior to D such that for any real c there exists $g \in C[D]$ such that for every compact neighborhood V of p_o,

$$\inf_{f \in \mathcal{F}} \|f - g\|_V > 0 \tag{2}$$

and such that $g(p_o) = c$. Then, $\mathcal{F}$ is nowhere dense in $C[D]$.

Proof: Let U be any non void open set in $C[D]$. We will produce a non void open subset of U that is disjoint from $\mathcal{F}$. Choose $G_o \in U$ and $r > 0$ such that $\|G - G_o\| < r$ implies $G \in U$. Let $c = G_o(p_o)$, and let g be the special function in $C[D]$ satisfying (2) whose existence is predicated. Let B be a closed ball in D, centered at p_o, such that

$$|G_o(p) - c| < r/3 \tag{3}$$

for all $p \in B$. Then, choose a ball B_o of smaller radius, also centered at p_o, such that

$$|g(p) - c| < r/3 \tag{4}$$

for all $p \in B_o$. Using Tietze, construct a continuous function G_1 on B such that

$$G_1(p) = \begin{cases} G_o(p) & \text{for } p \in \partial B \\ g(p) & \text{for } p \in B_o \end{cases}$$

and obeying

$$(5) \qquad |G_1(p) - c| < r/3$$

for all $p \in B$. Then, define $G_2(p)$ to be $G_o(p)$ off B and $G_1(p)$ on B. Observe that G_2 is continuous on D and agrees with G_o except on a small neighborhood of p_o where it has been modified to agree with the special function g locally. If $p \in B$, then

$$\begin{aligned}|G_2(p) - G_o(p)| &= |G_1(p) - G_o(p)| \\ &< |G_1(p) - c| + |c - G_o(p)| \\ &\leqq r/3 + r/3 < r \quad .\end{aligned}$$

Thus, $\|G_2 - G_o\|_D < r$ so that $G_2 \in U$.

Choose $\delta > 0$ so that $\|f - g\|_{B_o} > \delta$ for all $f \in$ and let

$$U_1 = \{\text{all } F \in U \text{ with } \|F - G_2\| < \delta/2\} \quad .$$

Suppose that $F \in U_1$ and $f \in \mathcal{F}$. Then,

$$\begin{aligned}\|f - F\|_D &\geqq \|f - F\|_{B_o} \\ &\geqq \|f - G_2\|_{B_o} - \|F - G_2\|_{B_o} \\ &\geqq \|f - g\|_{B_o} - \|F - G_2\|_{B_o} \\ &\geqq \delta - \delta/2 = \delta/2 \quad .\end{aligned}$$

Thus, U_1 is an open subset of U that contains G_2 and is disjoint from $\mathcal{F}$. Hence, $\mathcal{F}$ is nowhere dense in $C[D]$.

We now return to the nomographic functions and show that $\mathcal{N}^2$ is nowhere dense in $C[I^2]$, where $I = [-1, 1]$. We must exhibit functions g not in $\mathcal{N}^2$ that have the special property (2). We choose p_o to be $(0,0)$, and do not need to require the extra condition that $g(p_o) = c$ since $\mathcal{N}^2$ has the property that if $f \in \mathcal{N}^2$, so does $f + c$. For any $r > 0$, let V_r be the compact neighborhood of p_o consisting of those (x,y) with $|x| \leqq r$, $|y| \leqq r$.

Theorem 2. The function $g(x,y) = x^2 + xy + y^2 + 2x + y$ has the property that

$$r^2 > \inf_{f \in \mathcal{N}^2} \|f - g\|_{V_r} > r^3/10 \tag{6}$$

for all $r < .01$.

We begin by quoting one of the characterization theorems for nomographic functions, modified to match the notation and needs of the present proof. (See p. 293 of [2])

[Theorem 12] Let g be of class C' on the set $V_r = J^2$ where $J = [-r,r]$, and suppose that g_x and g_y are bounded below by $\sigma > 0$. Let $(u,v) = p_o = (0,0)$ and suppose that the distance from g to $\mathcal{N}^2$ in the space $C[V_r]$ is less than e, where $e < r\sigma/12$. Then, one of the following systems of inequalities must have a solution as indicated:

(i) For some choice of x_1 and x_2 in $[-r, r]$

$$\begin{aligned} |g(x_1, -r) - g(-r, 0)| &< 2e \\ |g(x_2, -r) - g(-r, r)| &< 2e \\ |g(x_1, r) - g(x_2, 0)| &< 2e \end{aligned} \tag{7}$$

(ii) For some choice of y_1 and y_2 in $[-r, r]$

$$\begin{aligned} |g(-r, y_1) - g(0, -r)| &< 2e \\ |g(-r, y_2) - g(r, -r)| &< 2e \\ |g(r, y_1) - g(0, y_2)| &< 2e \end{aligned} \tag{8}$$

If we apply this result to the function $g(x,y)$ given above, and set $x_i = rs_i$, $y_i = rt_i$, then if the distance from g to η^2 is less than ϵ, either there exist s_1 and s_2 with $|s_i| \leqq 1$ such that

$$|2s_1 + 1 + (s_1^2 - s_1)r| < 2\epsilon/r$$

$$(9) \qquad |2s_2 + (s_2^2 - s_2)r| < 2\epsilon/r$$

$$|2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r| < 2\epsilon/r$$

or there exist t_1 and t_2 with $|t_i| \leqq 1$ such that

$$|t_1 - 1 + (t_1^2 - t_1)r| < 2\epsilon/r$$

$$(10) \qquad |t_2 - 3 + (t_2^2 - t_2)r| < 2\epsilon/r$$

$$|t_1 - t_2 + 2 + (t_1^2 - t_2^2 + t_1 + 1)r| < 2\epsilon/r \quad .$$

To prove theorem 2, we now show that if $r < .01$ and $\epsilon = r^3/10$ then neither (9) nor (10) have a solution as specified. For (10) this is immediate. From the second inequality in (10) and the fact that $|t_i| \leqq 1$ we see that

$$|t_2 - 3| \leqq 2r + 2\epsilon/r = 2r + r^2/5$$

which would contradict $|t_2| \leqq 1$. To show that (9) also fails to have a solution, we proceed as follows. Set

$$A = 2s_1 + 1 + (s_1^2 - s_1)r$$

$$B = 2s_2 + (s_2^2 - s_2)r$$

$$C = 2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r$$

so that the desired inequalities are $|A| < r^2/5$, $|B| < r^2/5$ and $|C| < r^2/5$.

From the first, we obtain

$$|2s_1 + 1| \leqq 2r + r^2/5 < 3r$$

which shows that $s_1 = -1/2 + ar$, where $|a| \leqq 3/2$. Substituting this into A, we find that

$$|2a + 3/4| \leqq 3r + r/5 + \frac{9}{4} r^2 < 4r$$

so that $s_1 = -1/2 - (3/8)r + br^2$ where $|b| \leqq 2$. In the same way, starting from $|B| < r^2/5$, we find that $s_2 = cr^2$ where $|c| \leqq 1$.

Finally, observing that $C = A - B + (2s_1 + 1 - s_2)r$, we arrive at $|2s_1 + 1 - s_2| \leqq (3/5)r$ and substituting the values for s_1 and s_2. obtain

$$|-3/4 + (2b - c)r| \leqq 3/5$$

which clearly cannot hold if $r < .01$.

To obtain the left side of formula (6), we observe that the special function f_o defined by

$$f_o(x,y) = (x + y/2 + 1)^2 - 1$$

belongs to $\mathcal{N}^2$ and obeys $\|f_o - g\|_{V_r} < r^2$.

<u>Corollary</u>. $\mathcal{N}^2$ <u>is a nowhere dense subset of</u> $C[I^2]$.

To show now that this is also true for the class $\mathcal{N}^k$ of nomographic functions of k variables, for any $k > 2$, we will prove that one obtains no better nomographic approximation to the function $g(x,y) = x^2 + xy + y^2 + 2x + y$ by using functions $f \in \mathcal{N}^k$.

<u>Theorem</u> 3 <u>For any continuous function</u> $g(x_1, x_2)$ <u>of two variables</u>,

$$\inf_{f \in \mathcal{N}^k} \|f - g\|_{I^k} = \inf_{f^* \in \mathcal{N}^2} \|f^* - g\|_{I^2} .$$

<u>Proof</u>: Suppose that d is the distance in $C[I^k]$ from g to $\mathcal{N}^k$. Given $\delta > 0$, choose $f_o \in \mathcal{N}^k$ so that

$$d_o = \|f_o - g\|_{I^k} < d + \delta$$

and choose $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k) \varepsilon I^k$ so that

$$d_o = |f_o(\bar{x}) - g(\bar{x})|$$

$$= |h(\phi_1(\bar{x}_1) + \phi(\bar{x}_2) + \ldots + \phi_k(\bar{x}_k)) - g(\bar{x}_1, \bar{x}_2)| \quad .$$

Define a function ψ on I by

$$\psi(x_2) = \phi_2(x_2) + \phi_3(\bar{x}_3) + \ldots + \phi_k(\bar{x}_k)$$

and set

$$f^*(x_1, x_2) = h(\phi_1(x_1) + \psi(x_2)) \quad .$$

Then, $|f^*(\bar{x}_1, \bar{x}_2) - g(\bar{x}_1, \bar{x}_2)| = d_o = \|f_o - g\|_{I^k}$ while for every (x_1, x_2) in I^2

$$|f^*(x_1, x_2) - g(x_1,x_2)| = |f_o(x_1, x_2, \bar{x}_3, \ldots, \bar{x}_k) - g(x_1, x_2)|$$

$$\leq \|f_o - g\|_{I^k} \quad .$$

Accordingly, $\|f^* - g\|_{I^2} = \|f_o - g\|_{I^k} < d + \delta$ and since this holds for any $\delta > 0$,

$$\inf_{f \varepsilon \mathcal{M}^2} \|f - g\|_{I^2} \leq d = \inf_{f \varepsilon \mathcal{M}^k} \|f - g\|_{I^k} \quad .$$

Since $\mathcal{M}^2 \subset \mathcal{M}^k$, the reverse inequality also holds, proving the theorem.

It is known that four copies of $\mathcal{M}^2$ are not enough to give $C[I^2]$ as their algebraic sum. [4]. It would be of interest to know if the sum of four copies of $\mathcal{M}^2$ is dense in $C[I^2]$.

The argument used above will not suffice here since the special function $g(x,y)$ in fact already belongs to $\mathcal{M}^2 + \mathcal{M}^2$, for

$$g(x,y) = \{(x^2 - x - 6) + (y^2 - y)\} + \exp\{\log(x+2) + \log(y+3)\} \quad .$$

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approximations of functions of several variables; superpositions; nowhere dense sets

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A nomographic function of k variables is one that can be represented by the format

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where the ϕ_i and h are continuous. Any individual nomographic function is very special in nature, since it is constructed from functions of one variable and addition alone. However, Kolmogorov showed in 1957 that every continuous function of k variables has a representation as a sum of not more than $2k+1$

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ABSTRACT (continued)

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